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The CONV method for pricing options

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In this paper, we discuss a convolution based method, the CONV method, for pricing options with early-exercise features, in which the asset prices are modeled by Lévy processes.

When valuing and risk-managing exotic derivatives, practitioners demand fast and accurate prices and sensitivities. In this note we give an outline of a quadrature-based method for pricing options with early exercise features (the full paper can be found in [1]). By means of the risk-neutral valuation formula the price of any option without early exercise features can be written as an expectation of the discounted payoff of this option:

$$V(t, S(t)) = e^{-r\tau}\mathbb{E}\left[V(T, S(T))\right], \tag{1}$$

where V denotes the value of the option, r the risk-neutral interest rate, t the current time point, T the maturity of the option and $\tau = T - t$. The variable S denotes the asset on which the option contract is based. The expectation is taken with respect to the risk-neutral probability measure.

Bermudan options are options that can only be exercised at certain dates in the future. If the option is exercised at some time t_m the holder of the option obtains the exercise payoff $E(t, S(t))$. The Bermudan option price can then be found via,

$$\begin{cases} V(t_M, S(t_M)) &= E(t_M, S(t_M)) \\ C(t_m, S(t_m)) &= e^{-r\Delta t}\mathbb{E}_{t_m}\left[V(t_{m+1}, S(t_{m+1}))\right] \\ V(t_m, S(t_m)) &= \max\{C(t_m, S(t_m)), E(t_m, S(t_m))\}, \\ V(t_0, S(t_0)) &= C(t_0, S(t_0)), \end{cases} \quad m = M-1, \ldots, 1, \tag{2}$$

with C the continuation value and V the value of the option immediately prior to the exercise opportunity. The dynamic programming problem in (2) is a successive application of the risk-neutral valuation formula, as we can write the continuation value as

$$C(t_m, S(t_m)) = e^{-r\Delta t}\int_{-\infty}^{\infty} V(t_{m+1}, y) f(y|S(t_m))dy, \tag{3}$$

where $f(y|S(t_m))$ represents the probability density describing the transition from $S(t_m)$ at t_m to y at t_{m+1}. The main premise of the CONV method is that the conditional probability density $f(y|x)$ in (3) only depends on x and y via their difference $f(y|x) = f(y - x)$. After changing variables $z = y - x$, we take the Fourier transform,

$$\mathcal{F}\{h(t)\}(u) = \int_{-\infty}^{\infty} e^{iut}h(t)dt, \ \mathcal{F}^{-1}\{\hat{h}(u)\}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iut}\hat{h}(u)du, \tag{4}$$

of the resulting cross-correlation of the option value at time t_{m+1} and the density $f(z)$. If we dampen the continuation value by a factor $\exp(\alpha x)$, subsequently take its Fourier transform, change the order of integration and remember that $x = y - z$, we obtain

$$\begin{aligned} e^{r\Delta t}\mathcal{F}\{c(t_m, x)\}(u) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iuy}v(t_{m+1}, y)dy \; e^{-i(u-i\alpha)z}f(z)dz \\ &= \int_{-\infty}^{\infty} e^{iuy}v(t_{m+1}, y)dy\int_{-\infty}^{\infty} e^{-i(u-i\alpha)z}f(z)dz \\ &= \mathcal{F}\{e^{\alpha y}V(t_{m+1}, y)\}(u)\;\phi(-(u - i\alpha)). \end{aligned} \tag{5}$$

Small letters indicate damped quantities, i.e., $c(t_m, x) = e^{\alpha x}C(t_m, x)$ and $v(t_m, x + z) = e^{\alpha(x+z)}V(t_m, x + z)$. We also used the fact that the complex-valued Fourier transform of the density is the extended characteristic function

$$\phi(x + yi) = \int_{-\infty}^{\infty} e^{i(x+yi)z}f(z)dz. \tag{6}$$

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The essence of the CONV method is the calculation of a convolution

$$c(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \hat{v}(u) \phi\left(-(u - i\alpha)\right) du, \tag{7}$$

where $\hat{v}(u)$ is the Fourier transform of v. To be able to use the FFT means that we have to switch to logarithmic coordinates. For this reason the state variables x and y will represent $\ln S(t_m)$ and $\ln S(t_{m+1})$, up to a constant shift. We approximate the integrals appearing by a discrete sum, so that the FFT algorithm can be employed for their computation. This necessitates the use of uniform grids for u, x and y:

$$u_j = u_0 + j\Delta u,\ x_j = x_0 + j\Delta x,\ y_j = y_0 + j\Delta y, \tag{8}$$

where $j = 0, \ldots, N-1$. Though they may be centered around a different point, the x- and y-grids have the same mesh size: $\Delta x = \Delta y$. Further, the Nyquist relation must be satisfied, i.e.,

$$\Delta u \cdot \Delta y = \frac{2\pi}{N}. \tag{9}$$

Discretization yields

$$c(x_p) \approx \frac{\Delta u \Delta y}{2\pi} \sum_{j=0}^{N-1} e^{-iu_j x_p} \phi\left(-(u_j - i\alpha)\right) \sum_{n=0}^{N-1} w_n e^{iu_j y_n} v(y_n), \tag{10}$$

with integration weights w_n. Inserting the definitions of our grids into (10) yields:

$$c(x_p) \approx \frac{e^{-iu_0(x_0+p\Delta y)}}{2\pi} \Delta u \sum_{j=0}^{N-1} e^{-ijp2\pi/N} e^{ij(y_0-x_0)\Delta u} \phi\left(-(u_j - i\alpha)\right) \hat{v}(u_j), \tag{11}$$

where the Fourier transform of v is approximated by:

$$\hat{v}(u_j) \approx e^{iu_0 y_0} \Delta y \sum_{n=0}^{N-1} e^{ijn2\pi/N} e^{inu_0\Delta y} w_n v(y_n).$$

Let us now define the DFT and its inverse of a sequence $x_p,\ p = 0, \ldots, N-1$, as:

$$\mathcal{D}_j\{x_n\} := \sum_{n=0}^{N-1} e^{ijn2\pi/N} x_n, \quad \mathcal{D}_n^{-1}\{x_j\} = \frac{1}{N} \sum_{j=0}^{N-1} e^{-ijn2\pi/N} x_j,$$

and set $u_0 = -N/2\Delta u$. As $e^{inu_0\Delta y} = (-1)^n$ this finally leads us to:

$$c(x_p) \approx e^{iu_0(y_0-x_0)} (-1)^p \mathcal{D}_p^{-1}\{e^{ij(y_0-x_0)\Delta u} \phi\left(-(u_j - i\alpha)\right) \mathcal{D}_j\{(-1)^n w_n v(y_n)\}\}.$$

The resulting CONV method converges very fast and is flexible w.r.t. different underlying processes. Furthermore, it can be used for pricing American options, or some exotic options, like basket, barrier and Asian options.

References

[1] R. Lord, F. Fang, F. Bervoets and C.W. Oosterlee, A Fast and Accurate FFT-Based Method for Pricing Early-Exercise Options under Lévy Processes. SSRN, page http://ssrn.com/abstract=966046, 2007. Submitted for publication.